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On hydromagnetic Rossby waves excited by travelling forcing effects

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The pattern and propagation of hydromagnetic Rossby waves excited by travelling forcing effects on a rotating spherical shell of incompressible, inviscid, perfectly conducting fluid are studied using Lighthill's technique. The basic magnetic field H_0 is assumed to be uniform and acting in the 'beta-plane' in an arbitrary direction. The two situations when the forcing effects travel along and perpendicular to H_0 are considered.

The steady forcing effects travelling in the direction of an eastward or westward H_0 excite two types of wave systems. The first consists of unattenuated signals directly behind or ahead of the forcing effect. The other system consists of semicircular waves travelling in all directions or waves travelling in a limited wedge.

When the forcing effect moves eastward, perpendicular to an H_0, acting northward, the waves excited in the steady case are confined to certain wedges and trail behind with cusp-shaped wave crests. The magnetic field increases the semi-angle of the wedges, so that the region of disturbance is expanded. An oscillatory forcing effect generates various systems of waves. If the frequency of oscillation exceeds a certain critical frequency, excitement of waves in all directions is possible. The situation with a westward-moving forcing effect is also discussed. The effect of large rotation is to reduce the length of the waves.

1. Introduction

The study of the magnetohydrodynamics of rotating fluids is a very recent development in fluid mechanics and has applications in geophysics and astrophysics. Some of the features of MHD phenomena in rotating fluids can be understood by studying plane waves in uniformly rotating incompressible fluids. Hide (1966, 1969), Braginskiy (1967, 1970), Malkus (1967), Stewartson (1967) and Gans (1971), among others, have studied the hydromagnetic wave modes in incompressible rotating fluids.

In this paper, we consider a rotating spherical shell of inviscid, incompressible, perfectly conducting fluid. We introduce an approximation similar to the one introduced by Rossby and Haurwitz in the study of planetary-scale tidal oscillations in the atmosphere. This type of approximation to the true spherical surface metric by a flat one is called the 'β-plane' approximation. A uniform magnetic field H_0 acts in the β-plane in a direction making an angle θ with the eastward direction measured in the positive sense. Using Lighthill's (1967) technique, we analyse the pattern and propagation of Rossby waves generated by steady or oscillatory forcing effects moving with a uniform velocity U in the direction of H_0 or perpendicular to it. These are the waves in systems of variable Coriolis parameter, the motion being purely horizontal with zero divergence. Such solenoidal motions can be expected to be realized in experiments with spherical shells of liquid like those of Fultz & Long (1951) and Frenzen (1955). Lighthill (1967) applied his technique to examine Rossby waves, excited by a moving wind-stress pattern in a β-plane ocean, whose pattern corresponds to the non-magnetic case of the present study of hydromagnetic Rossby waves.

In §3 we consider the case when the forcing effect travels in the direction of H_0, i.e. makes the same angle θ as H_0 does with the eastward direction. An eastward-moving forcing effect ($\theta = 0$) generates semicircular waves of wavelength $2\pi[(V_A^2 - U^2)/U\beta]^{\frac{1}{2}}$, where $U < V_A$, V_A being the Alfvén wave speed. These semicircular waves travel ahead of the forcing region in an arbitrary direction. Further, signals are found directly ahead, and consist of the disturbance integrated in the west–east direction and subjected to a 'high-pass filter' with respect to its north–south wavenumber components. The disturbances found directly behind are subjected to the complementary 'low-pass filter' although they do include some high wavenumber components. If $U > V_A$ unattenuated disturbances trail directly behind the forcing region. The situation with a westward-moving forcing effect ($\theta = \pi$) is discussed in detail in §3.2. The waves generated by a forcing effect moving in an intermediate direction ($0 < \theta < \pi$) are also identified. The case of oscillatory forcing effects is discussed in §3.4. As the frequency of oscillation increases, there is directional spread of the waves.

In §4 we consider the situation when the forcing effect moves eastward, perpendicular to an H_0 acting northward. The waves in the steady case are confined to a wedge whose semi-angle depends on $A^2 = V_A^2/U^2$, and the point of inflexion on each wavenumber curve corresponds to waves on the boundary of such a wedge. The waves trail behind the forcing effect and have cusp-shaped wave crests inside the wedge. The effect of the magnetic field is to increase the semi-angle of the wedge, i.e. to expand the region of disturbance. If the forcing effect oscillates with a frequency σ_0, the pattern depends critically on the parameter $N = \sigma_0(U\beta)^{-\frac{1}{2}}$. If N exceeds 2 the waves are found all round the forcing effect. The wavenumber curves for a steady westward-moving forcing effect consist of two split branches passing through the points $(\pm 1, 0)$ in the wavenumber plane. The waves corresponding to the two branches become superposed and trail behind the forcing effect.

In §5 we consider the effect of large rotation on Rossby waves. It is found that the wavelength of the waves is reduced by a factor $R_0^{\frac{1}{2}}$, where

$$R_0 = V_A^2/U\Omega^* R',$$

R' being the mean radius of the spherical shell and $\Omega^* = 2\Omega\cos\phi_0$, with ϕ_0 the mean latitude. Finally, geophysical applications of the results are discussed in §6.

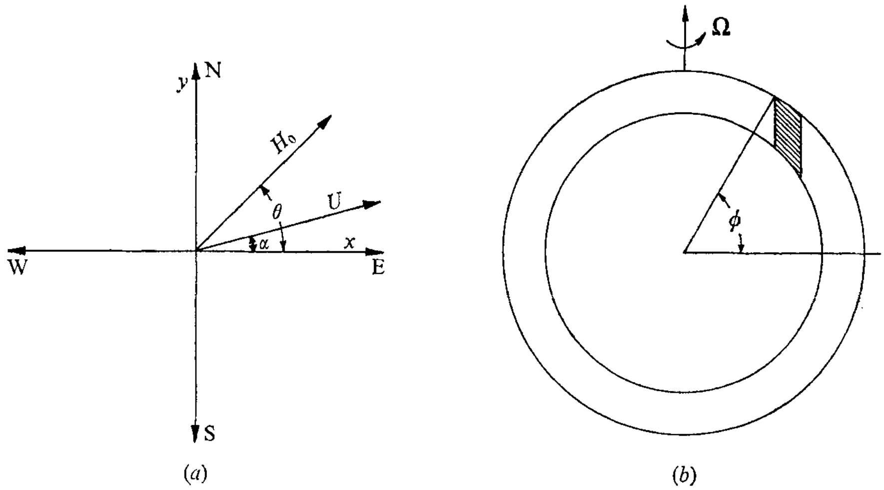

FIGURE 1. Co-ordinate system.

2. Formulation and formal solution

We use a local Cartesian frame whose origin is located at the mean latitude ϕ_0 of the region under consideration, the x, y and z axes being eastward, northward and upward respectively (see figure 1 (*a*)). For the horizontal Coriolis acceleration we take $\Omega \sin\phi$, the vertical component of $\mathbf{\Omega}$, the angular velocity of rotation, where ϕ is the angle of latitude (see figure 1 (*b*)). The Coriolis parameter

$$f^* \equiv 2\Omega \sin\phi$$

is expressed in the form $f^* = f_0 + \beta y$, where $f_0 = 2\Omega \sin\phi_0$ and

$$\beta = 2{\cdot}3 \times 10^{-13} \cos\phi_0 \, \mathrm{cm^{-1}\,s^{-1}}.$$

A magnetic field perpendicular to the x, y plane has no effect on the two-dimensional problem; hence, without loss of generality we restrict our attention to the case of a magnetic field whose lines of force are parallel to the x, y plane. So, we treat the case of an impressed uniform magnetic field $\mathbf{H}_0 = H_0 \cos\theta \mathbf{i} + H_0 \sin\theta \mathbf{j}$ acting in a direction θ measured in the positive sense from the eastward direction. The linearized equations of motion of two-dimensional hydromagnetic flow of inviscid, incompressible, perfectly conducting fluid in the β-plane can be written as

$$\frac{\partial u_x}{\partial t} - f u_y = -\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(H_0 \cos\theta \frac{\partial}{\partial x} + H_0 \sin\theta \frac{\partial}{\partial y}\right) h_x, \tag{1}$$

$$\frac{\partial u_y}{\partial t} + f u_x = -\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\left(H_0 \cos\theta \frac{\partial}{\partial x} + H_0 \sin\theta \frac{\partial}{\partial y}\right) h_y, \tag{2}$$

$$\frac{\partial h_x}{\partial t} = \left(H_0 \cos\theta \frac{\partial}{\partial x} + H_0 \sin\theta \frac{\partial}{\partial y}\right) u_x, \tag{3}$$

$$\frac{\partial h_y}{\partial t} = \left(H_0 \cos\theta \frac{\partial}{\partial x} + H_0 \sin\theta \frac{\partial}{\partial y}\right) u_y, \tag{4}$$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0, \quad \frac{\partial h_x}{\partial x} + \frac{\partial h_y}{\partial y} = 0, \tag{5}$$

where u_x and u_y are the velocity components relative to the rotating frame of reference, h_x and h_y are the magnetic field components, p is the hydrodynamic pressure divided by the density ρ and μ is the magnetic permeability. We can show that the relative vorticity $\zeta = \partial u_y/\partial x - \partial u_x/\partial y$ satisfies the equation

$$\left[\frac{\partial^2}{\partial t^2} - V_A^2\left(\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}\right)^2\right]\nabla_H^2\zeta+\beta\frac{\partial}{\partial t}\frac{\partial}{\partial x}\zeta = 0, \tag{6}$$

where ∇_H^2 is the two-dimensional Laplace operator and $V_A^2 = \mu H_0^2/\rho$ the square of Alfvén wave speed. The effect of the forcing region can be incorporated in the governing differential equation (6) by replacing the right-hand side by a non-zero forcing term

$$e^{-i\sigma_0 t}f(\mathbf{r}-\mathbf{U}t), \tag{7}$$

where $\mathbf{r} = x\mathbf{i}+y\mathbf{i}$ is the position vector and $\mathbf{U} = U\cos\alpha\mathbf{i}+U\sin\alpha\mathbf{j}$, i.e. the forcing effect moves with uniform velocity U in a direction making an angle α with the x axis (see figure 1 (a)). The differential equation admits a plane-wave solution of the type

$$\zeta = \zeta_0\exp\{i(-\sigma t+lx+my)\}$$

if the dispersion relation

$$S(\sigma_0, l, m) \equiv [\{Ul\cos\alpha+Um\sin\alpha+\sigma_0\}^2 - V_A^2\{l\cos\theta+m\sin\theta\}^2](l^2+m^2) + \beta l[Ul\cos\alpha+U\sin\alpha+\sigma_0] = 0 \tag{8}$$

is satisfied (since $\sigma = \sigma_0+Ul\cos\alpha+Um\sin\alpha$ owing to the Doppler effect). If $f(\mathbf{r})$ vanishes outside a finite region, by taking Fourier transforms a formal solution of (6) with (7) as its right-hand side may be obtained as

$$\zeta = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{F(\mathbf{k})\exp[i\{-\sigma_0 t+\mathbf{k}.(\mathbf{r}-\mathbf{U}t)\}]}{S(\sigma_0+\mathbf{U}.\mathbf{k}, l, m)}\,dl\,dm, \tag{9}$$

where

$$f(\mathbf{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(\mathbf{k})\,e^{i\mathbf{k}.\mathbf{r}}\,dl\,dm$$

and $\mathbf{k} = l\mathbf{i}+m\mathbf{j}$ is the wavenumber vector. A method of obtaining a unique solution (satisfying a radiation condition) of integral equation (9) is explained in Lighthill (1960, 1967). In wavenumber (l, m) space, at each point of the curve $S(\sigma_0) = 0$ we draw an arrow normal to the curve, choosing from the two normal directions the one pointing towards the curve $S(\sigma_0+\delta) = 0$ with δ small and positive. That is, the arrow is in the direction of σ increasing. The waves (if any) found in some particular direction stretching out from the forcing region are those given by a point on the wavenumber curve whose arrow is in that particular direction. The amplitude of the waves generated by the forcing term is asymptotically given by

$$\left[\frac{(2\pi)^{\frac{3}{2}}}{|k|^{\frac{1}{2}}R^{\frac{1}{2}}}\right]\left[\frac{F(\mathbf{k})}{\nabla S(\sigma_0)}\right], \tag{10}$$

where $R = |\mathbf{r}-\mathbf{U}t|$ means distance from the forcing region, ∇ is the operator grad with respect to (l, m) space and k is the curvature of the wavenumber curve. A straight portion of the wavenumber curve generates waves without attenuation, the first factor in (10) being replaced by 2π. In the case of multiply covered straight parts of the wavenumber curve, the method of obtaining the amplitude is to be modified as in §3.1.

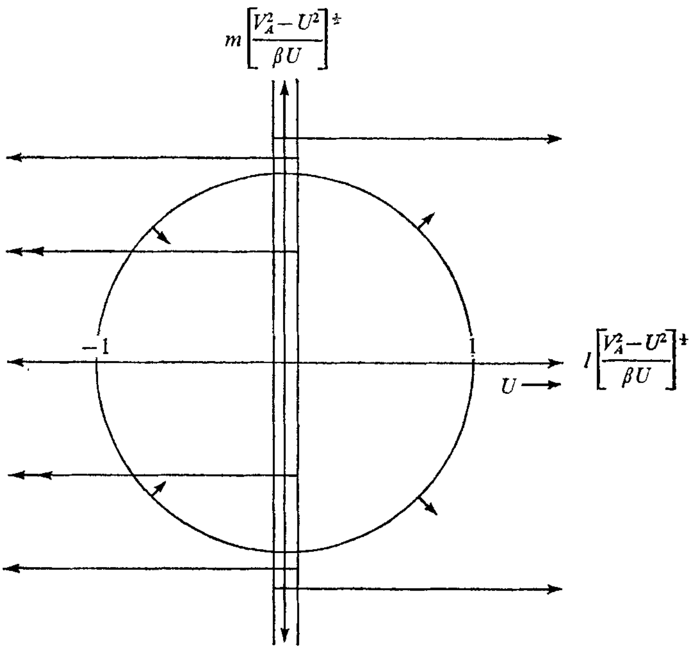

FIGURE 2. Wavenumber curves for hydromagnetic Rossby waves generated by a steady forcing effect travelling eastward with velocity $(U, 0)$.

3. Waves excited by forcing effects travelling in the direction of H_0

In this section we shall examine Rossby waves excited by a steady or oscillatory forcing effect moving in the direction of H_0. Since the directions of U and H_0 are the same, we have $\alpha = \theta$ in the dispersion relation (8). Thus

$$S(\sigma_0, l, m) = [\{Ul\cos\theta + Um\sin\theta + \sigma_0\}^2 - V_A^2\,(l\cos\theta + m\sin\theta)^2]\,(l^2+m^2) + \beta l(Ul\cos\theta + Um\sin\theta + \sigma_0] = 0. \quad (11)$$

We shall discuss the dispersion relation (11) for different values of θ. The three cases that arise for $U > V_A$, $U = V_A$ and $U < V_A$ are treated for each value of θ in the following subsections.

3.1. *Steady eastward-moving forcing effect* ($\theta = 0$)

For a steady forcing effect moving eastward with uniform velocity $(U, 0)$, relation (11) with $\theta = 0$ takes the form

$$S(\sigma_0, l, m) = (U^2 - V_A^2)\,l^2\,(l^2+m^2) + \beta U l^2 = 0. \quad (12)$$

Case (i). $U < V_A$. The wavenumber curve in this case is a circle:

$$(l^2+m^2) = \beta U/(V_A^2 = U^2). \quad (13)$$

The wavenumbers corresponding to the points on this circle correspond to waves of uniform wavelength $2\pi[(V_A^2 - U^2)/U\beta]^{\frac{1}{2}}$ propagating in an arbitrary direction. The directions of the arrows are as shown in figure 2. The waves, which have circular wave crests, travel ahead of the forcing effect, filling the eastward-facing hemisphere ahead of it. In addition, the curve (12) contains a straight portion, the line $l = 0$ taken twice. The arrows along the appropriate normal must be drawn on the two straight lines and the normal directions appropriate to each line may or may not coincide. By drawing $S(Ul+\delta, l, m) = 0$ for small positive δ, we observe that the line $l = 0$ splits into two parts. The arrows on these parts point to the west if

$$m < [U\beta/(V_A^2 - U^2)]^{\frac{1}{2}}. \tag{14}$$

Otherwise, the arrows on one part point to the crest and on the other part they point to the east. In other words, disturbances independent of x and t are possible ahead of the forcing region, if the transverse wavenumber $|m|$ exceeds

$$[U\beta/(V_A^2 - U^2)]^{\frac{1}{2}},$$

whereas, if $|m|$ does not exceed $[U\beta/(V_A^2 - U^2)]^{\frac{1}{2}}$, these disturbances trail behind. The physical explanation is that those waves with zero phase velocity whose wave crests are parallel to the east–west direction have group velocities

$$C_1 = \frac{-\beta + (\beta^2 + 4V_A^2 m^4)^{\frac{1}{2}}}{2m^2}, \quad C_2 = \frac{-\beta - (\beta^2 + 4V_A^2 m^4)^{\frac{1}{2}}}{2m^2} \tag{15}$$

directed to the east or west. The inequality (14) can be written as

$$U - C_1 > 0. \tag{16}$$

When (16) is not satisfied C_1 exceeds U, so that forward influence becomes possible; otherwise, U exceeds all group velocities, so that all the disturbances trail behind. The waves propagate without attenuation because the associated part of the wavenumber curve is a straight line. The disturbance that travels ahead (eastward) of the forcing region is the transverse disturbance created by the forcing effect modified by a 'high-pass filter' passing only wavenumbers above $[\beta U/(V_A^2 - U^2)]^{\frac{1}{2}}$. The disturbance extending to the west is not subjected merely to the complementary 'low-pass filter'; it also includes some high wavenumbers.

To estimate the magnitude of the above unattenuated disturbances, the method leading to (10) cannot be used unchanged because the integral to be estimated has a double-pole singularity on the doubly covered portion of the wavenumber curve. With S as in (11) and σ_0 replaced by $i\epsilon$ and θ by zero, (9) becomes

$$\zeta = \int_{-\infty}^{\infty} e^{imy}\, dm \int_{-\infty}^{\infty} \frac{F(l,m)\exp\{i(x-Ut)\,l\}}{[(Ul+i\epsilon)^2 - V_A^2 l^2]\,(l^2+m^2) + \beta l(Ul+i\epsilon)}\, dl \tag{17}$$

and the probelm is to estimate the inner integral when $|x - Ut|$ is large. When ϵ is positive but very small, the double pole at $l = 0$ splits into two simple poles at

$$l_1 = \frac{-i\epsilon}{U - C_2}, \quad l_2 = \frac{-i\epsilon}{U - C_1}. \tag{18}$$

When (14) is not satisfied these are on opposite sides of the real axis, so that by Jordan's lemma there is a contribution to the inner integral from the pole at $l = l_2$ when $x - Ut$ is positive and from $l = l_1$ when it is negative; but, when (14) is satisfied, both l_1 and l_2 have negative imaginary parts and so there is no contribution at all for $(x - Ut) > 0$; this agrees with the direction of the arrows in figure 2.

The contribution to the inner integral of (17) from the residue at the pole at $l = l_1$ is

$$\frac{2i\pi\,[F(l,m)\exp\{i(x-Ut)\,l\}]}{[(U^2-V_A^2)\,m^2+U\beta]\,(l_1-l_2)} \tag{19}$$

and the contribution at the pole $l = l_2$ can be written similarly. The difficulty in taking the limit as $\epsilon \to 0$ disappears for the following reason. If steady forcing terms $g_x(x-Ut,y)$ and $g_y(x-Ut,y)$ are added to the right-hand sides of (1) and (2), the differential equation for ζ becomes

$$\left[\frac{\partial^2}{\partial t^2}-V_A^2\left(\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}\right)^2\right]\nabla_H^2\zeta+\beta\frac{\partial}{\partial t}\frac{\partial}{\partial x}\zeta = \frac{\partial}{\partial t}\left[\frac{\partial}{\partial x}\nabla_H^2 g_y-\frac{\partial}{\partial y}\nabla_H^2 g_x\right]. \tag{20}$$

For the steady disturbance the right-hand side of (20) becomes

$$U\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\nabla_H^2 g_x-\frac{\partial}{\partial x}\nabla_H^2 g_y\right]. \tag{21}$$

The Fourier transform of (21) contains a factor l and hence vanishes at $l = 0$. Then, taking the limit as $\epsilon \to 0$ of (19) and a similar expression for l_2 by L'Hospital's rule, we find the residues to be

$$\frac{4\pi\, im^2\,[\partial F/\partial l]_{l=0}}{(\beta^2+4V_A^2m^4)^{\frac{1}{2}}\,[(\beta^2+4V_A^2m^4)^{\frac{1}{2}}\pm(2Um^2+\beta)]} \tag{22}$$

with the upper sign applying in the limit for $l = l_1$ and the lower sign in the limit for $l = l_2$. Disturbances are found ahead of the forcing region only when (14) is not satisfied. Further, we see that the forward disturbances are merely l_2 disturbances for which negative sign is taken in (22). We see that the amplitude of these l_2 disturbances progressively decays as $|m|-[U\beta/(V_A^2-U^2)]^{\frac{1}{2}}$ increases. We find that a powerful forward disturbance is excited even when the group velocity only slightly exceeds U although there is no forward disturbance at all below $[U\beta/(V_A^2-U^2)]^{\frac{1}{2}}$. The physical reason for this is explained by Lighthill (1967): a forcing effect of given extent can generate a forward-moving wave component most powerfully when its group velocity only slightly exceeds the speed of travel of the forcing effect because the time available before the wave component escapes from the forcing region is then the greatest. Increase of (22) to extremely large values would be restricted by dissipation by nonlinearity, or by the finite duration of the forcing effect.

Case (ii). $U > V_A$. In this case, the wavenumber curve consists only of two coincident straight lines. The arrows on both the parts l_1 and l_2 point to the west, so that the waves trail behind without attenuation.

3.2. *Westward-moving forcing effect* ($\theta = \pi$)

The dispersion relation for a westward-moving forcing effect is obtained by changing U to $-U$ in (12), i.e.

$$(U^2-V_A^2)\,(l^2+m^2)\,l^2-\beta Ul^2 = 0. \tag{23}$$

Case (i). $U > V_A$. Figure 3 shows that the wavenumber curve consists of a circle

$$(l^2+m^2) = \beta U/(U^2-V_A^2) \tag{24}$$

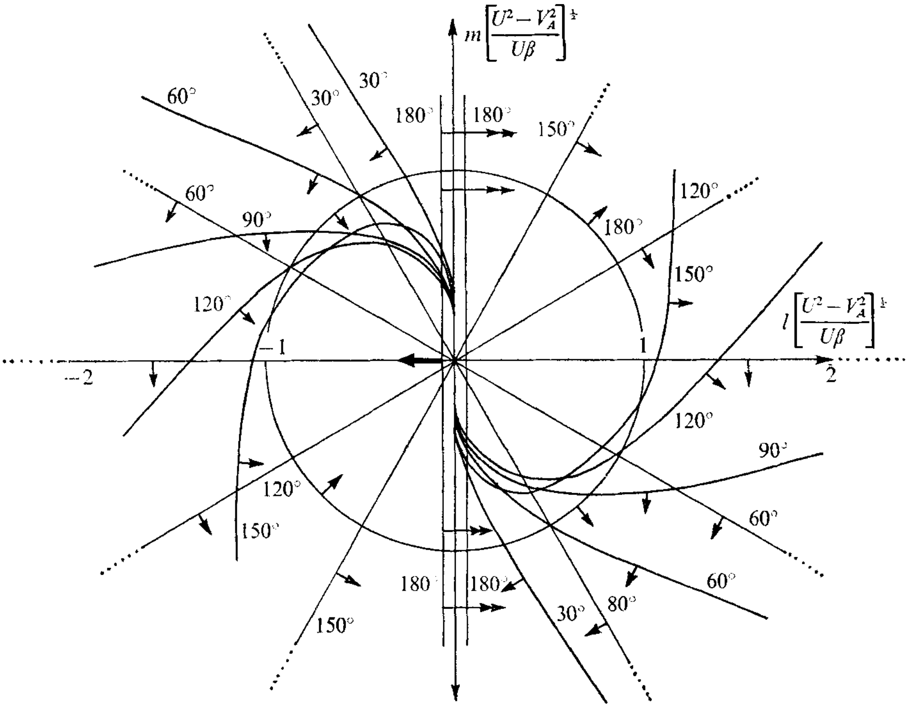

FIGURE 3. Wavenumber curves for hydromagnetic Rossby waves generated by steady forcing effects travelling with uniform velocity U in directions making positive angles $\alpha = 30°, 60°, 90°, 120°, 150°$ and $180°$ (marked on each curve) with the eastward direction; ···, asymptotes.

and $l = 0$ taken twice. The waves satisfying (24), which have circular wave crests, trail behind the forcing region. These waves are of uniform wavelength $2\pi[(U^2 - V_A^2)/\beta U]^{\frac{1}{2}}$ and travel in an arbitrary direction. By drawing the curve $S(-Ul+\delta, l, m) = 0$ for small positive δ, we find that the line $l = 0$ splits into two parts:

$$l_1' = \frac{i\dot{\epsilon}}{U-C_1}, \quad l_2' = \frac{i\dot{\epsilon}}{U-C_2}, \tag{25}$$

where C_1 and C_2 are given by (15). Both l_1' and l_2' disturbances trail behind the forcing region for all transverse wavenumbers.

Case (ii). $U < V_A$. The wavenumber curve consists only of two straight portions $l = 0$ taken twice which split into two parts l_1' and l_2' given by (25). We find that the l_2' disturbances trail behind the forcing region for all transverse wavenumbers whereas l_1 disturbances trail behind or travel ahead of the forcing region according as $m \gtrless [U\beta/(V_A^2 - U^2)]^{\frac{1}{2}}$. If $U = V_A$, both l_1' and l_2' disturbances trail behind the forcing effect.

3.3 *Steady forcing effect moving in an arbitrary direction* $(0 < \theta < \pi)$

In this case θ takes arbitrary values such that $0 < \theta < \pi$. We consider a forcing effect moving with uniform velocity $(U\cos\theta,\ U\sin\theta)$. The dispersion relation (11) then becomes

$$(U^2 - V_A^2)\,[l\cos\theta + m\sin\theta]^2\,(l^2+m^2) + \beta Ul(l\cos\theta + m\sin\theta) = 0. \tag{26}$$

Case (i). $U > V_A$. The wavenumber curve consists of the cubic curve

$$(U^2 - V_A^2)(l\cos\theta + m\sin\theta)(l^2+m^2) + \beta U l = 0 \tag{27}$$

and a straight-line part $$l\cos\theta + m\sin\theta = 0. \tag{28}$$

The wavenumber curves satisfying (27) and (28) for $\alpha = 30°$, $60°$, $90°$, $120°$ and $150°$ are shown in figure 3. As is indicated by the arrows, the waves corresponding to points on (27) are found in a wedge defined by the negative-l axis and a line which makes an angle θ measured in the positive sense with the negative-l axis. If θ is relatively small ($\theta < \frac{1}{2}\pi$) the disturbances propagate behind the forcing region. If $\theta > \frac{1}{2}\pi$ the disturbances are found behind as well as ahead of the forcing effect. The point of inflexion on each curve at the origin, where the arrow points westward for all values of θ, means that, whenever the forcing term has significant wavenumber components in this region, a strong signal will be found to the west of the disturbance. The amplitudes of these disturbances attenuate like $R^{-\frac{1}{3}}$ instead of $R^{-\frac{1}{2}}$. In addition, we have unattenuating disturbances trailing directly behind the forcing region for all values of θ.

Case (ii). $U < V_A$. In this case the dispersion relation (26) can be written as

$$(V_A^2 - U^2)[l\cos\theta + m\sin\theta]^2(l^2+m^2) - \beta U l[l\cos\theta + m\sin\theta] = 0. \tag{29}$$

Thus, the wavenumber curve consists of the cubic

$$(V_A^2 - U^2)(l\cos\theta + m\sin\theta)(l^2+m^2) - \beta U l = 0 \tag{30}$$

and the straight portion (28). The cubic curve (30) for each θ is a reflexion of the cubic curve shown in figure 3 for $\pi-\theta$ in the m axis. The waves corresponding to the points satisfying (30) are found in a wedge formed by the negative-l-axis and a line making an angle $\pi-\theta$ measured in the negative sense with the negative-l axis. If $\theta < \frac{1}{2}\pi$ the waves are travelling behind as well as ahead of the forcing region and if $\theta > \frac{1}{2}\pi$, the waves are found only ahead of the forcing region. The unattenuated disturbances travel directly ahead of the forcing effect. The strong signal to the west is still found in this case if there are significant wavenumbers about the origin, which is a point of inflexion on each curve.

Case (iii). $U = V_A$. The wavenumber curve consists of two straight portions given by $l = 0$ and $l\cos\theta + m\sin\theta = 0$ for each θ. In every case, the transverse disturbances trail directly behind the forcing effect and the waves corresponding to the points on $l = 0$ travel westward.

3.4. *Travelling oscillatory forcing effects*

In this section we study the waves generated by an oscillatory forcing effect travelling with uniform velocity $(U\cos\theta,\ U\sin\theta)$. The dispersion relation is given by (11) and the wavenumber curves are drawn in figure 4 for various σ_0 when $\theta = 30°$, taking the case $U > V_A$. For $\theta < \frac{1}{2}\pi$ we have found in § 3.3 that a steady forcing effect generates waves in a limited wedge trailing behind it. It is the purpose of this section to examine whether or not this limitation holds for the oscillatory case.

First we notice that the unattenuated disturbances excited in the steady case are eliminated. The straight portion (28) and the cubic curve (27) are replaced by a curve which splits into two branches. From figure 4 we see that, if $m < 0$,

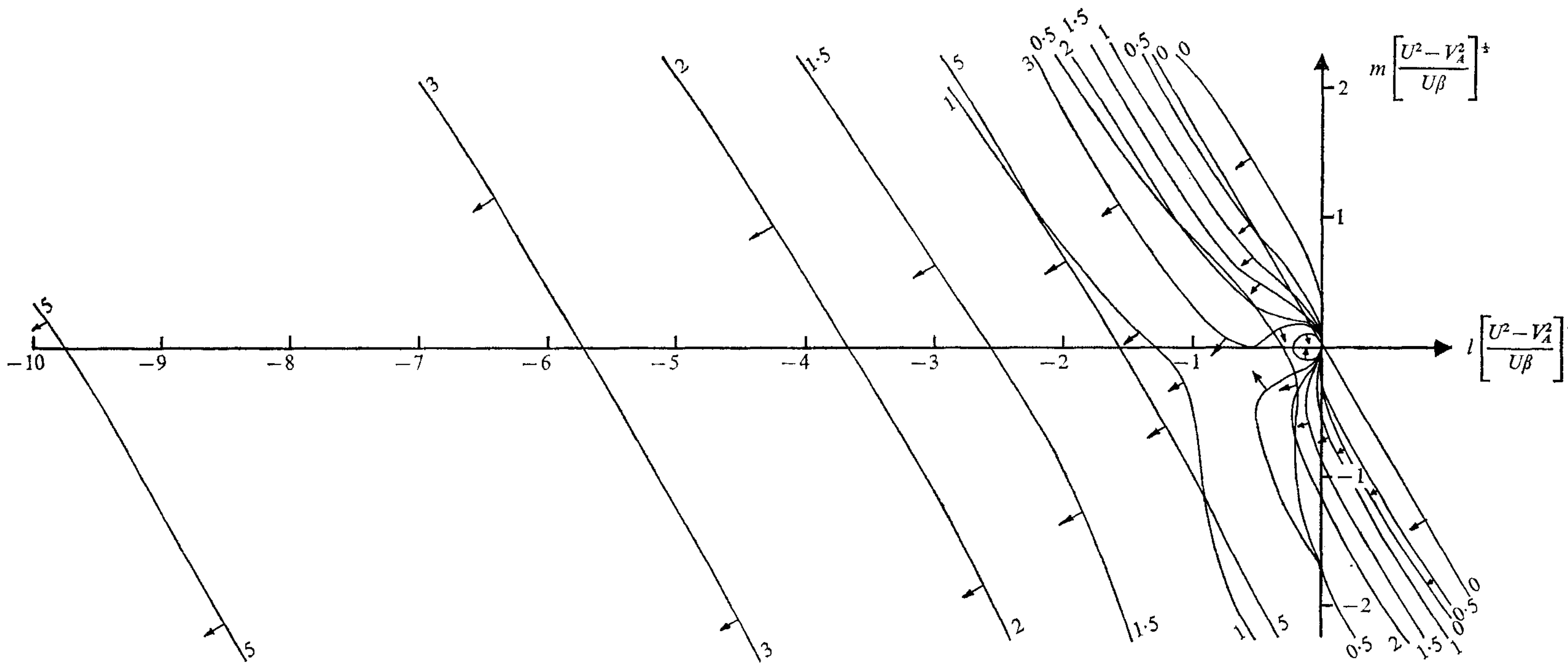

FIGURE 4. Wavenumber curves for hydromagnetic Rossby waves generated by an oscillatory forcing effect travelling with uniform velocity $U > V_A$ in a direction making a positive angle 30° with the eastward direction. The number marked on each curve is the value of

$$L = 2\sigma_0\left(\frac{U^3\beta}{U^2-V_A^2}\right)^{-\frac{1}{2}}$$

where σ_0 is the frequency.

the branch passing through the origin corresponds to (28) and the branch away from the origin corresponds to (27). If $m > 0$, the reverse is true. The shape of $S(\sigma_0)$ depends critically on the value of the frequency parameter

$$L = 2\sigma_0 [U^3\beta/(U^2 - V_A^2)]^{-\frac{1}{2}}. \quad (31)$$

For $L < 2$ the trailing character of the waves persists, but on the other hand, any components with $L > 2$ have a much greater directional spread. There exists a critical value of $L = L_1$ (which is seen to lie between 3 and 4) beyond which large changes take place. For $L = 5$ we find that the wavenumber curve splits into three branches. The waves corresponding to the ring-shaped branch passing through the origin are found practically all around the forcing region. However, only waves corresponding to very small wavenumbers, say with

$$[(l^2+m^2)\{(U^2 - V_A^2)/U\beta\}]^{\frac{1}{2}} < \tfrac{1}{2},$$

will be found all round the forcing effect. The other two branches become nearly straight and hence the waves corresponding to the points on them continue to trail behind the forcing region.

4. Waves excited by forcing effects moving perpendicular to H_0

In this section, we discuss the pattern and propagation of Rossby waves produced by a steady or oscillatory forcing effect moving in a direction perpendicular to H_0. We have taken the forcing effect, for definiteness, to be moving eastward and H_0 acting northward. The dispersion relation (8) becomes

$$S(\sigma_0, l, m) = [(Ul+\sigma_0)^2 - V_A^2 m^2](l^2+m^2) + \beta l(Ul+\sigma_0) = 0. \quad (32)$$

If the transformation $l' = l\,(U/\beta)^{\frac{1}{2}}, m' = m(U/\beta)^{\frac{1}{2}}$ is introduced (32) becomes on dropping the primes

$$S(\sigma_0) = [(N+l)^2 - A^2m^2](l^2+m^2) + l\,(N+l) = 0, \quad (33)$$

where N is the frequency parameter given by $N = \sigma_0(U\beta)^{-\frac{1}{2}}$ and $A^2 = V_A^2/U^2$.

4.1. *Steady forcing effect*

For $N = 0$, the dispersion relation for a steady forcing effect takes the form

$$S(0) = (l^2 - A^2m^2)(l^2+m^2) + l^2 = 0. \quad (34)$$

The wavenumber curves satisfying (34) are drawn for $A^2 = \frac{1}{2}$, 1 and 5 in figure 5. The curves are symmetric about the l and m axes and are asymptotic to the lines $m = \pm l/A$. The arrows on the curve indicate that the waves trail behind the forcing region for all values of A^2. In contrast with the situation in § 3, the waves trail behind in all three cases $U > V_A$, $U = V_A$ and $U < V_A$. In other words, if U is fixed, the magnetic field strength has no effect on the trailing character of the waves.

We can see from figure 5 that, as $A^2 \to 0$, the two branches of the wavenumber curve on each side of the m axis coincide with it. This wake-like disturbance has been pointed out by Lighthill (1967) and was noticed by Fultz & Long (1951) in their experiments.

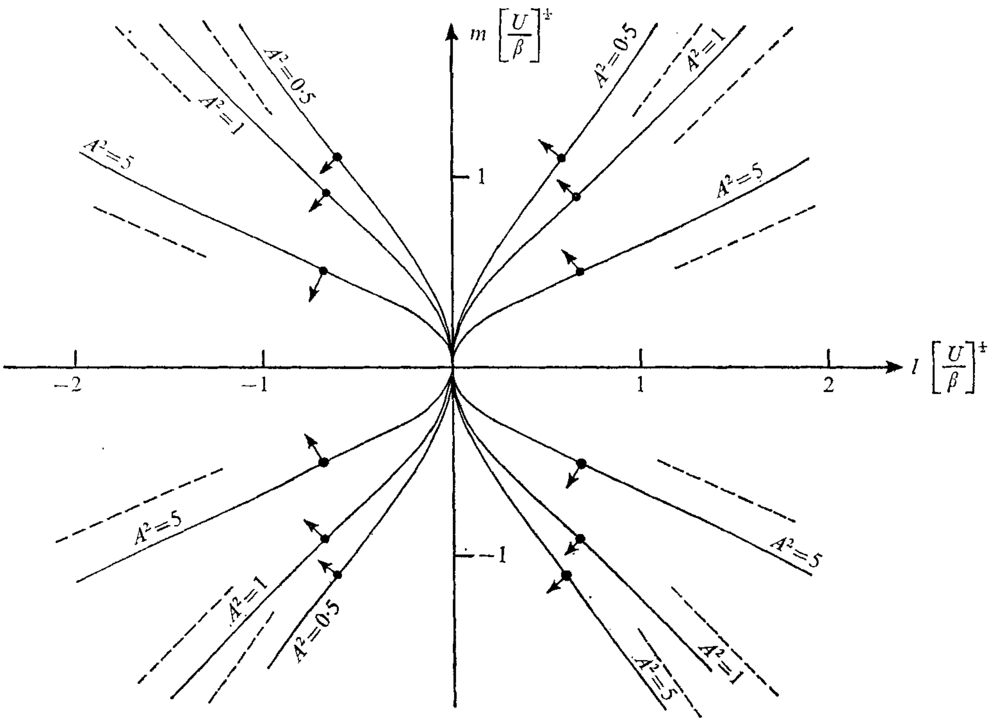

FIGURE 5. Wavenumber curves for hydromagnetic Rossby waves generated by a steady forcing effect moving with uniform velocity U in the eastward direction for $A^2 = 0{\cdot}5$, 1 and 5. – –, asymptotes; ●, points of inflexion.

The point of inflexion $(\lambda\cos\xi,\ \lambda\sin\xi)$ on the wavenumber curve is given by $\lambda = Z(A^2-Z^2)^{-\frac{1}{2}}$, where $Z = \cot\xi$ satisfies the equation

$$(1-A^2)Z^6-6A^2Z^4+3A^2(A^2-1)Z^2+2A^4 = 0. \qquad (35)$$

Equation (35) is a cubic in Z^2 whose positive real roots give the points of inflexion. They are shown by a solid circle on the wavenumber curves for $A^2 = 0{\cdot}5$, 1 and 5 in figure 5. The arrows on the wavenumber curves at the points of inflexion make the maximum angle ψ with the eastward direction measured in the positive sense, where

$$\tan\psi = \frac{Z[2A^2+Z^2(A^2-1)]}{Z^4+A^2}. \qquad (36)$$

This implies that all the waves generated at the forcing effect propagate westward, confining themselves in the wedges of semi-angle ψ, and the decay of their amplitudes is given by (10). It may be noted that the waves corresponding to the point of inflexion attenuate like $R^{-\frac{1}{3}}$ instead of $R^{-\frac{1}{2}}$. It is seen that $|\tan\psi|$ will increase as A^2 is increased. This means that the wedge in which the waves are confined expands as the strength of the magnetic field is increased.

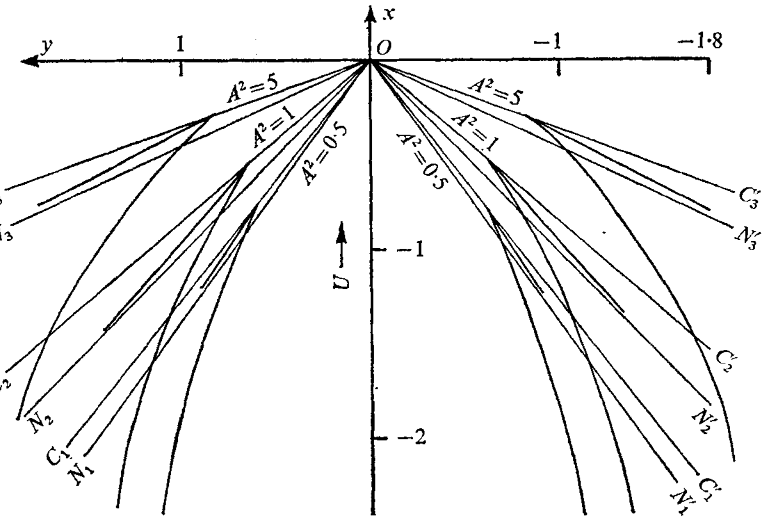

FIGURE 6. Curves of constant phase for wavenumber curves shown in figure 5 for $A^2 = 0{\cdot}5$, 1 and 5.

4.2. *Curves of constant phase*

The curves of constant phase, which are polar reciprocals of $S(\sigma_0) = 0$, are the loci of points (x, y) defined by

$$\left.\begin{aligned} x &= -M^2\left[\frac{2(N+l)(l^2+m^2)+2\{(N+1)^2-A^2m^2\}l+N+2l}{|2(l^2+m^2)\{(N+l)l-A^2-m^2\}-Nl|}\right] \\ &\qquad\qquad \times \operatorname{sgn}[2(N+1)(l^2+m^2)+1], \\ y &= -M^2\left[\frac{2m\{(N+l)^2-A^2m^2\}-2A^2m(l^2+m^2)}{|2(l^2+m^2)\{(N+l)l-A^2m^2\}-Nl|}\right] \\ &\qquad\qquad \times \operatorname{sgn}[2(N+l)(l^2+m^2)+l], \end{aligned}\right\} \quad (37)$$

where M is the constant value of the phase $\mathbf{k}.\mathbf{r}$. For the steady case, i.e. $N = 0$, the curves of constant phase are drawn for $A^2 = 0{\cdot}5$, 1 and 5 in figure 6. They contain cusps corresponding to the point of inflexion on the wavenumber curve. The curves of constant phase in the second and third quadrants in figure 6 correspond to the wavenumber curves in the first and fourth quadrants (or in the third and second quadrants) in figure 5 respectively. This is indicated by the direction of the arrows on the wavenumber curves. The waves are confined to the wedges (OC_1, OC_1'), etc., where OC_1 and OC_1' are the lines of cusps. The waves corresponding to the points of inflexion propagate along the lines of cusps and the waves corresponding to the other points of $S(0) = 0$ propagate inside the wedge. The waves of shorter wavelength propagate inside OC_1, ON_1, etc. The waves of longer wavelength propagate outside ON_1, etc., where ON_1, etc., are normals to the asymptotes of the wavenumber curves in the direction of the arrows. It is seen from figure 6 that the wedges open up as A^2 is increased, which confirms the result that the effect of the magnetic field is to increase the angle of the wedge in which the waves are confined.

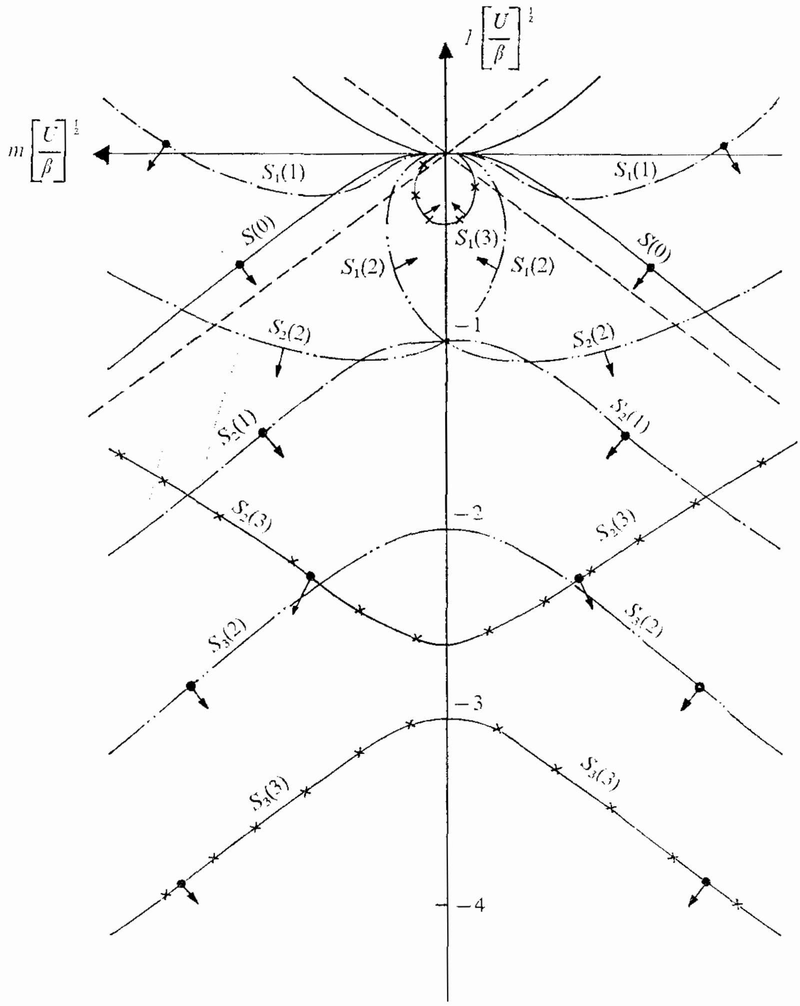

FIGURE 7. Wavenumber curves for hydromagnetic Rossby waves generated by an oscillatory forcing effect moving in the uniform velocity $(U, 0)$. In $S_i(j)$, i denotes the branch number and j denotes the values of $N = \sigma_0(U/\beta)^{-\frac{1}{2}}$, $i = 0$ and $j = 0$ corresponding to the steady case, $A^2 = 0{\cdot}5$. ●, points of inflexion.

4.3. *An oscillatory forcing effect*

If the forcing effect is oscillatory, the dispersion relation (33) contains a non-zero parameter N in addition to A^2, which is fixed as $0{\cdot}5$ in the present discussion. For all N the wavenumber curves are symmetric about the l axis and are asymptotic to the lines $m = \pm l/A$. We consider the waves generated for $N > 2$, $N = 2$ and $N < 2$ in turn as $N = 2$ is a critical value where the transition to a new pattern and propagation occurs.

For $N < 2$, we have two branches of the wavenumber curve, one passing through the origin and the other passing through the point $(-N, 0)$. The wave number curves for $N = 0, 1, 2$ and 3 are shown in figure 7. The waves are found to fill the wedges formed by the normals at the points of inflexion, which are shown as solid circles on the branches of the wavenumber curves. For $N = 1$ the waves corresponding to the points on the branch $S_1(1)$ are confined to the wedge of semi-angle 37·6° and those corresponding to the points on $S_1(2)$ fill the wedge of semi-angle 38·7°.

For $N = 2$ the wavenumber curves drawn in figure 7 consists of three branches, $S_1(1)$, $S_2(2)$ and $S_3(2)$. The waves corresponding to the points on $S_1(2)$ are found all round the forcing region except in a wedge of semi-angle 18·4° ahead of it. The waves corresponding to the points on $S_2(2)$ and $S_3(2)$ are confined to wedges of semi-angle 54° and 38° respectively.

For $N > 2$ the wavenumber curve consists of three distinct branches $S_1(3)$, $S_2(3)$ and $S_3(3)$. These are drawn in figure 7 for $N = 3$. The branch $S_1(3)$ is an oval one involving very small wavenumbers, say with $\{(l^2+m^2)(U/\beta)\}^{\frac{1}{2}}$ less than 1. If the forcing effect generates components with such small wavenumbers, the waves are found all round it. The waves corresponding to the branch $S_2(3)$ are confined to a wedge of semi-angle 66·25° and those corresponding to the points on $S_3(3)$ are confined to wedge of semi-angle 26·35°. All the waves generated by the forcing effect trail behind it and are confined to their respective wedges. The general pattern and propagation of the waves are not affected by the hydromagnetic parameter A^2, which only affects the semi-angles of the wedges.

4.4. *Steady westward-moving forcing effects*

The dispersion relation for a steady westward-moving forcing effect is obtained by changing the sign of the second term in (34). We therefore have

$$(l^2-A^2m^2)(l^2+m^2)-l^2 = 0. \tag{38}$$

The wavenumber curves for $A^2 = 0{\cdot}5$, 1 and 5 are shown in figure 8. The curves split into two branches $S_{\pm}(A^2)$ passing through the points $(\pm 1, 0)$ respectively. The curves are symmetric about the l axis and tend asymptotically to the lines $m = \pm l/A$. The points of inflexion are given by (35). There are no points of inflexion on the wavenumber curves for $A^2 > 1$. The waves corresponding to the points on the wavenumber curves $S_+(1)$ and $S_+(5)$ trail behind the forcing region inside the wedges formed by the normals to the asymptotes drawn in the direction of the arrows. Hence, the waves corresponding to the points on $S_+(1)$ and $S_+(5)$ are found inside the wedges of semi-angles 45° and 66°, respectively.

There is a point of inflexion on the wavenumber curve $S_+(0{\cdot}5)$ shown by a solid circle in figure 8. The curves of constant phase $F_+(0{\cdot}5)$ of $S_+(0{\cdot}5)$ are drawn in figure 9. We find that $F_+(0{\cdot}5)$ has a cusp corresponding to the point of inflexion. ON and ON' are normals to the asymptotes and OC, OC' are the lines of cusps. The waves corresponding to the point of inflexion on $S_+(0{\cdot}5)$ propagate along these lines of cusps. The waves of shorter wavelength are confined to the wedge (OC, OC') of semi-angle 8°. The waves of longer wavelength propagate outside this wedge but confined to the wedge (ON, ON') of semi-angle 36°. For any particular direction, there are three waves propagating inside the wedge (OC, OC').

We notice that, as the magnetic field strength is increased, the semi-angles of the wedges are also increased because the inclination of the normals to the corresponding asymptotes to the eastward direction is increased. As $A^2 \to 0$,

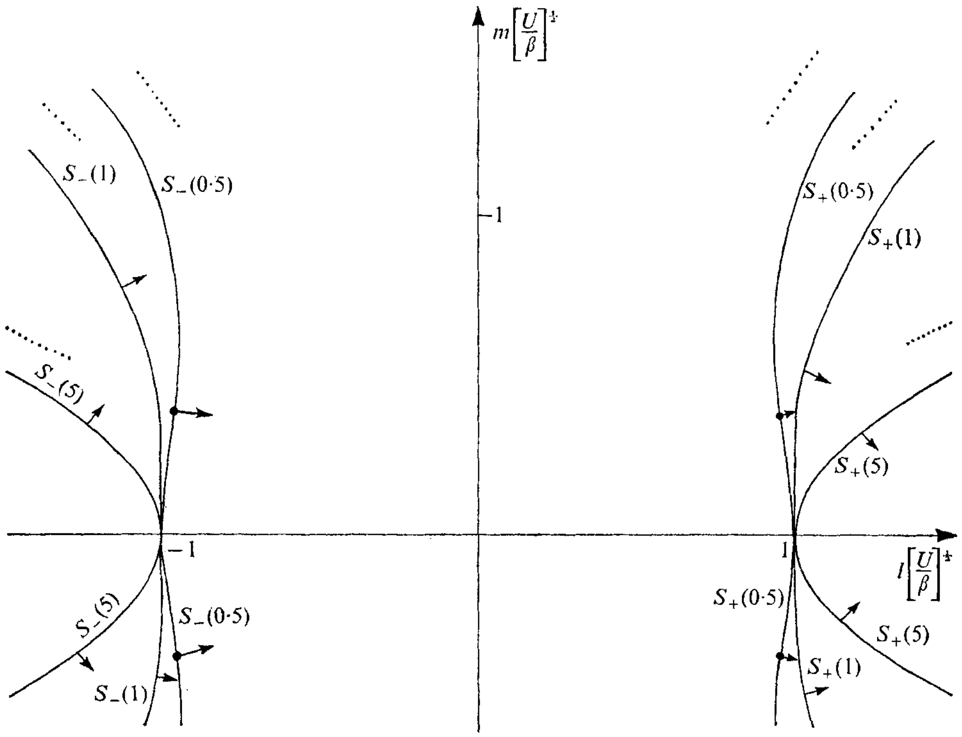

FIGURE 8. Wavenumber curves for hydromagnetic Rossby waves generated by a steady forcing effect travelling with uniform velocity $(-U, 0)$. $S_{\pm}(A^2)$ denotes the branches passing through $(\pm 1, 0)$; $A^2 = 0{\cdot}5$, 1 and 5.

the two branches $S_{\pm}(A^2)$ close up to become a circle and the line $l = 0$. This case was discussed by Lighthill (1967).

Finally, we consider the presence of the branch $S_-(A^2)$ in figure 8.† Even though there is symmetry with respect to the origin the present problem is an exception to the general statement that the solution which satisfies the radiation condition takes only one out of each pair $\pm\mathbf{k}$ (Lighthill 1960, p. 407). For it follows that, if the wavenumber curve for a particular frequency is a closed curve not enclosing singularities, the arrows will either be always outwards from the curve or always inwards. It is in such cases that, owing to the symmetry with respect to the origin, where the arrows at $\mathbf{k}$ and $-\mathbf{k}$ will lie in opposite directions, the above general statement applies and hence only one of them contributes. However, in the present example the wavenumber curve is not closed and it includes a singularity, namely the one at the origin. This makes it possible for the frequency to increase inwards on one part of the wavenumber curve and outwards on another part. As a result the arrows at $\pm\mathbf{k}$ lie in the same direction and hence the two waves corresponding to $S_{\pm}(A^2)$ are superposed in that direction and their amplitudes are given by (10).

† Private communication from Mr G. V. Prabhakara Rao.

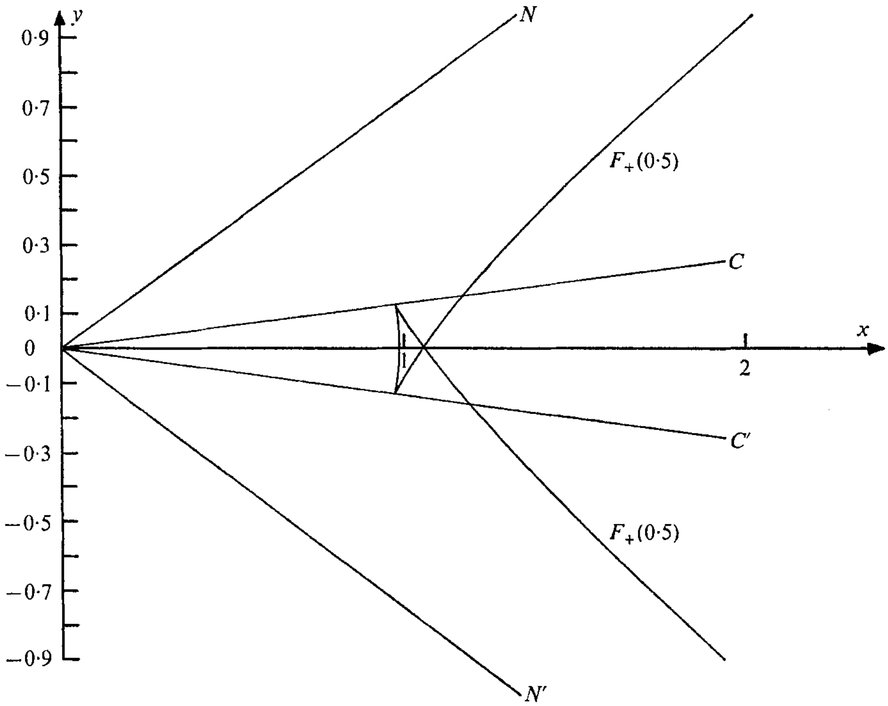

FIGURE 9. Curves of constant phase $F_+(0{\cdot}5)$ for the wavenumber curve $S_+(0{\cdot}5)$.

5. Effect of large rotation

In the study of magnetohydrodynamic rotating fluids the key parameter is the Rossby number

$$R_s = V_A/L\Omega$$

defined with respect to the Alfvén wave speed V_A (Hide 1971; Gans 1971), where $R_s \ll 1$ corresponds to strong rotation. In the geophysical context $R_s \ll 1$ is important (Hide 1971). Hence, we shall proceed to examine the effects of large rotation on Rossby waves. We introduce a similar parameter

$$R_0 = V_A^2/R'\Omega^* U,$$

where $\Omega^* = 2\Omega\cos\phi_0$ and R' is the mean radius of the spherical shell, so that $R_0 \ll 1$ corresponds to large rotation. In the steady case, the non-dimensional form of (32) is

$$(l_1^2 - A_1^2 m_1^2)(l_1^2 + m_1^2) + R_0^{-1} l_1^2 = 0, \tag{39}$$

where $l_1 = R'l$ and $m_1 = R'm$. We find that the pattern and propagation of the waves corresponding to the points of the wavenumber curves represented by (34) and (39) are identical; but the wavelengths of the waves corresponding to the points on the wavenumber curve (39) are reduced by a factor $R_0^{\frac{1}{2}}$. Hence, large rotation has the effect of reducing the wavelength of Rossby waves.

6. Concluding remarks

Hide (1966) considered the model of a rotating spherical shell of inviscid, incompressible, perfectly conducting fluid in order to examine the free hydromagnetic oscillations in the earth's corre. The liquid metallic core of the earth occupies a very nearly spherical shell and the main geomagnetic field arises in the core. Transmission of magnetic energy between different parts of the core is mainly accomplished by the hydromagnetic waves. There are various types of forcing effects like the release of gravitational energy within the core. Hence, the present study of hydromagnetic Rossby waves is of geophysical interest. It may be noted that the hydromagnetic Rossby waves exhibit a tendency to propagate to the west of the spherical shell in most of the situations discussed in §§ 3 and 4. This tendency indicates the transport of magnetic energy towards the west.

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